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Heat transfer at high Péclet number from a small sphere freely rotating in a simple shear field

By ANDREAS ACRIVOS
Department of Chemical Engineering, Stanford University

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The problem of heat transfer at high Péclet number P_e from a sphere freely rotating in a simple shear field is considered theoretically for the case of small shear Reynolds numbers. It is shown that the present problem is in many respects similar to that of heat transfer past a freely rotating cylinder which was recently solved by Frankel & Acrivos (1968). By taking advantage of the close analogy between these two problems, an approximate method of solution is developed according to which the asymptotic Nusselt number for $P_e \to \infty$ is 9, i.e. $4\frac{1}{2}$ times its value for pure conduction. As in the corresponding case of the cylinder, the fact that the asymptotic Nusselt number is independent of P_e results from the presence of a region of closed streamlines which completely surrounds the rotating sphere.

1. Introduction

As shown theoretically by Frankel & Acrivos (1968), the Nusselt number N_u for heat transfer from an isothermal cylinder freely rotating in a low Reynolds number shear flow becomes independent of the magnitude of the impressed shear, and equal to 5·73, for asymptotically large Péclet numbers P_e. This result, confirmed experimentally by Robertson & Acrivos (1970), is in contrast to the corresponding case of a particle held fixed where, as is well known, N_u becomes $O(P_e^{\frac{1}{3}})$ for $P_e \gg 1$. Of course, this seemingly paradoxical difference between the two cases can easily be explained by noting that, at high P_e, a freely rotating cylinder is surrounded by a region of effectively isothermal streamlines across which heat can be transferred by conduction alone, whereas, when the cylinder is stationary relative to the main stream, the transport of heat takes place, both by conduction and convection, across a conventional thermal boundary layer of thickness $O(P_e^{-\frac{1}{3}})$. Nevertheless, the dissimilarity between the corresponding asymptotic expressions for the Nusselt number is significant, because it serves to emphasize that, when the Péclet number is large, the rate of heat transfer from a particle to a surrounding fluid depends rather critically on whether the streamlines near the heated surface are open or closed. Thus, in dealing with phenomena of this type, whether theoretically or experimentally, it is important that a clear distinction be made between the two general categories referred to above.

So far, attention has been directed primarily to the case of an infinite circular cylinder in a simple shear, because the two-dimensionality of the flow field simplified both the theory and the experiments (Frankel & Acrivos 1968; Robertson & Acrivos 1970). We now turn to the corresponding problem of heat transfer from a small sphere which, of course, is of more than academic interest in view of its relevance to possible physical applications.

As shown by Cox, Zia & Mason (1968), who presented a detailed theoretical analysis supported by experimental results, the low Reynolds number shear flow past a rotating cylinder, and that past a sphere, have many points in common. In particular, in both cases the rotating surface is surrounded by a set of closed streamlines all of which are contained within the space enclosed by a limiting streamline, for the cylinder, and a limiting stream surface, for the sphere. Thus, one would expect the corresponding heat transfer problems also to be similar, especially so at high Péclet numbers where the temperature field is closely related to the structure of the streamlines near the rotating body.

2. Basic equations

Consider then the familiar energy equation in its dimensionless form,

$$\mathbf{u}.\nabla T = \frac{1}{P_e}\nabla^2 T, \tag{2.1}$$

where $\mathbf{u}$ is the velocity vector divided by Sa, a is the radius of the cylinder or the sphere in terms of which all lengths are rendered dimensionless, S is the magnitude of the shear far from the body, T is the dimensionless temperature, and P_e is the Péclet number $R\sigma$ with R being the shear Reynolds number Sa^2/ν and σ the Prandtl number. The boundary conditions are that

$$T = 1 \quad \text{at} \quad r = 1, \qquad T = 0 \quad \text{as} \quad r \to \infty.$$

It is evident from (2.1) that, in the absence of any large temperature gradients, $\mathbf{u}.\nabla T \to 0$ as $P_e \to \infty$, hence the streamlines become isothermal. However, to obtain the temperature distribution it is necessary to take heat conduction into account since, as mentioned earlier, this is the primary mode of heat transfer at high P_e when the streamlines are closed. To this end, we multiply both sides of (2.1) by dt, t denoting the time, and integrate along a closed streamline. Thus, since T is single valued,

$$\oint \nabla^2 T\, dt = P_e \oint (\mathbf{u}.\nabla T)\, dt \equiv P_e \oint (dT/dt)\, dt = 0, \tag{2.2}$$

for all closed streamlines and for all values of P_e.

The above general result simplifies in the case of a two-dimensional flow and $P_e \gg 1$ (Pan & Acrivos 1968) into

$$\frac{d}{d\psi}\Gamma(\psi)\frac{dT}{d\psi} = \text{constant}, \tag{2.3}$$

where ψ denotes the stream function and $\Gamma(\psi)$ the circulation along a given ψ. Actually, (2.3) represents nothing more than a heat balance, for it merely states that the net rate of heat conduction across any closed streamline must be constant from one streamline to the next, an obvious requirement. Nevertheless, (2.3) suffices to determine uniquely the temperature field, given the velocity distribution and, therefore, $\Gamma(\psi)$ (Pan & Acrivos 1968; Frankel & Acrivos 1968).

For a three-dimensional flow, the problem becomes much more complicated. To be sure, as shown by Grimshaw (1969), one can again derive formally an integral condition, similar to (2.3), for the net heat transport across the isothermal stream surface; however, this expression cannot lead to any quantitative results because, in contrast to the two-dimensional case where the isotherms coincide with the streamlines, the location of these stream surfaces is *a priori* unknown. Therefore, it is necessary to return to (2.2) even though the latter does not represent a heat balance, but merely an integral constraint arising from the single-valuedness of the temperature field.

Let us now consider specifically the case of a small sphere freely rotating in the shear field given by, in dimensionless form,

$$u_1 = u_2 = 0, \quad u_3 = x_2. \tag{2.4}$$

As shown by Cox *et al.* (1968), the streamlines are formed by the intersection of the two sets of surfaces

$$x_1 = Crf(r), \quad x_2 = \pm rf(r)\,[E+g(r)]^{\frac{1}{2}}, \tag{2.5}$$

where C and E are parameters, $r \equiv [x_1^2+x_2^2+x_3^2]^{\frac{1}{2}}$, and

$$f(r) \equiv (r^3 - \tfrac{5}{2} + \tfrac{3}{2}r^{-2})^{-\frac{1}{3}}, \quad g(r) \equiv \int_r^\infty y^{-3} f(y)\,dy. \tag{2.6}$$

Furthermore, both f and g are monotonically decreasing functions of r and have the following properties:

(*a*) as $r \rightarrow 1$

$$\left.\begin{aligned} f &= (\tfrac{2}{15})^{\frac{1}{3}}\,(r-1)^{-\frac{2}{3}}\{1+\tfrac{2}{9}(r-1)+O(r-1)^2\}, \\ g &= g(1) - 3(\tfrac{2}{15})^{\frac{1}{3}}\,(r-1)^{\frac{1}{3}}\{1-\tfrac{25}{36}(r-1)+O(r-1)^2\}; \end{aligned}\right\} \tag{2.7}$$

(*b*) as $r \rightarrow \infty$

$$\left.\begin{aligned} f &= r^{-1}+\tfrac{5}{6}r^{-4}+O(r^{-6}), \\ g &= \tfrac{1}{3}r^{-3}+\tfrac{5}{36}r^{-6}+O(r^{-8}). \end{aligned}\right\} \tag{2.8}$$

Therefore, it is easily seen from the above and (2.5) that all streamlines are open if $E > 0$, that all of them are closed if $-g(1) \leqslant E \leqslant 0$, and that no streamline can exist if $E < -g(1)$. Moreover, when $-g(1) \leqslant E \leqslant 0$, C must lie between $-C^*$ and $+C^*$, where

$$C^* f(r^*) = 1 \quad \text{with} \quad E+g(r^*) = 0, \quad -g(1) \leqslant E \leqslant 0. \tag{2.9}$$

Finally, from (2.5) and (2.7) we can readily deduce that $E = -g(1)$ corresponds to the surface of the sphere and that all the closed streamlines are contained within the space lying between the sphere and the *limiting* three-dimensional stream surface $E = 0$.

As remarked earlier, the temperature along any given closed streamline approaches a constant value as $P_e \rightarrow \infty$. In addition, since E and C are constant along any streamline, it evidently follows that the temperature T becomes a function only of E and C. This result, together with (2.2), will now be used to obtain an equation for the temperature distribution in the limit $P_e \rightarrow \infty$.

Let $x_1 = r\cos\theta$, $x_2 = r\sin\theta\cos\phi$, $x_3 = r\sin\theta\sin\phi$. Then, in view of (2.5), one can, in principle at least, express r and θ in terms of ϕ, E and C. Consequently, since $T = T(E, C)$,

$$\nabla^2 T = b_1 \frac{\partial^2 T}{\partial E^2} + b_2 \frac{\partial^2 T}{\partial E\,\partial C} + b_3 \frac{\partial^2 T}{\partial C^2} + b_4 \frac{\partial T}{\partial E} + b_5 \frac{\partial T}{\partial C}, \tag{2.10}$$

where the b's are functions of ϕ, E and C. Therefore, since

$$dt = r\sin\theta\, d\phi / u_\phi,$$

with u_ϕ denoting the velocity component along ϕ, (2.2) together with (2.10) becomes

$$0 = \frac{1}{4\pi}\int_0^{2\pi} \nabla^2 T \frac{r\sin\theta}{u_\phi} d\phi = \bar{b}_1 \frac{\partial^2 T}{\partial E^2} + \bar{b}_2 \frac{\partial^2 T}{\partial E\,\partial C} + \bar{b}_3 \frac{\partial^2 T}{\partial C^2} + \bar{b}_4 \frac{\partial T}{\partial E} + \bar{b}_5 \frac{\partial T}{\partial C}, \tag{2.11}$$

where, of course, the integration is along the streamline (E, C). The $\bar{b}$'s appearing above are now functions only of E and C, hence (2.11) is a partial differential equation for the temperature distribution inside the region of closed streamlines.

The boundary conditions associated with (2.11) are

$$T = 1 \quad \text{at} \quad E = -g(1), \quad \text{the surface of the sphere,}$$
$$T = 0 \quad \text{at} \quad E = 0, \quad \text{the limiting surface,}$$

and that T be finite within the space $-g(1) \leqslant E \leqslant 0$, $-C^* \leqslant C \leqslant C^*$, where C^* is given by (2.9). The requirement that the limiting surface be at the free-stream temperature, although perhaps rather obvious, can be proved rigorously by a method identical to that used previously (Frankel & Acrivos 1968) to show that, for the corresponding case of the cylinder, the temperature along the limiting streamline equals that of the free stream as $P_e \to \infty$.

The quantity of most interest here is the Nusselt number N_u which, based on the diameter $2a$, takes on the form

$$N_u = \frac{1}{2\pi}\int_0^{\pi}\int_0^{2\pi} -\left(\frac{\partial T}{\partial r}\right)_{r=1} \sin\theta\, d\theta\, d\phi. \tag{2.12}$$

Hence, for pure conduction ($P_e = 0$), $N_u = 2$. Moreover, in view of (2.11) the Nusselt number will also become independent of P_e when $P_e \to \infty$, although one would certainly expect an asymptotic value greater than 2. Thus, as anticipated in the introduction, there exists a close analogy, of which we shall presently take advantage, between the corresponding heat transfer problem of the cylinder and that of the sphere.

Evidently, it is impossible to obtain an exact solution of (2.11) except, perhaps, by numerical means. This is so because, first of all, (2.11) is a rather complicated partial differential equation, in contrast to (2.3), where T is a function of the single variable ψ. In addition, the coefficients $\bar{b}_k(E, C)$ can only be calculated numerically following a tedious integration of (2.10). And, finally, the domain over which (2.11) applies in the (E, C) plane has the rather irregular boundaries given by (2.9). It is for these reasons then that an approximate technique was devised whose success depends on the close analogy between the cylinder and sphere problems referred to above.

3. An approximate value for the asymptotic Nusselt number

Let us return to (2.3) again. It has the form of the familiar one-dimensional heat conduction equation with a variable thermal conductivity, and its exact solution can readily be obtained. For the purpose of attacking the corresponding sphere problem, however, we shall construct an approximate solution which is based on the well-known fact that the rate of heat transfer in systems having variable properties can generally be accurately estimated using only their values near the heated surface.

Consider then the form of $\Gamma(\psi)$ for the freely rotating cylinder near $\psi = 0$, the surface of the cylinder. At low Reynolds numbers (e.g. Robertson & Acrivos 1970),

$$\psi = \tfrac{1}{4}(r^2-1) - \tfrac{1}{4}(r-r^{-1})^2\cos 2\phi, \tag{3.1}$$

hence

$$\Gamma(\psi) \equiv \int_0^{2\pi}\left(\frac{u_r^2+u_\phi^2}{u_\phi}\right) r\,d\phi, \tag{3.2}$$

where u_r, u_ϕ and r are determined as functions of ψ and ϕ from (3.1). Although $\Gamma(\psi)$ cannot be represented in closed form, it is possible to show that

$$\frac{\Gamma}{\pi} = 1 + 4\psi - 16\psi^2 + \tfrac{256}{3}\psi^3 + O(\psi^4). \tag{3.3}$$

Accordingly, using $\Gamma(\psi)$ as given by (3.3), we wish to solve (2.3), subject to the boundary conditions

$$T = 1 \quad \text{at} \quad \psi = 0, \quad T = 0 \quad \text{at} \quad \psi = \tfrac{1}{4} \text{ (the limiting streamline).}$$

Of course, this can be accomplished in a variety of ways; however, for the purpose of applying the technique to the sphere problem we proceed as follows:

Let

$$T = 1 - a_1\psi + a_2\psi^2 - a_3\psi^3 + \ldots, \tag{3.4}$$

which, because of (2.3) and (3.3), leads to the recursion relations

$$a_2 = 2a_1, \quad a_3 = \tfrac{32}{3}a_1, \quad \text{etc.} \tag{3.5}$$

Next, we consider the result of truncating the series (3.4) at successively higher terms and applying the boundary condition $T = 0$ at $\psi = \frac{1}{4}$. Thus retaining only two terms of (3.4), we have that

$$T = 1 - a_1\psi,$$

therefore $a_1^{(1)} = 4$, where $a_1^{(1)}$ denotes the first approximation to a_1. If three terms are retained and use is made of the well-known Euler transformation to improve the convergence of (3.4) as $\psi \to \frac{1}{4}$, then

$$1 = \tfrac{1}{2}\psi\{a_1^{(2)} + \tfrac{1}{2}(a_1^{(2)} - a_2^{(1)}\psi) + \ldots\} = \tfrac{1}{8}(\tfrac{3}{2}a_1^{(2)} - \tfrac{1}{8}a_2^{(1)}),$$

where $a_1^{(2)}$ is obtained from (3.5) using the previously computed value of a_1, i.e. $a_1^{(1)}$. Consequently, $a_1^{(2)} = 6$.

When three terms are retained

$$1 = \tfrac{1}{8}(\tfrac{7}{4}a_1^{(3)} - \tfrac{1}{4}a_2^{(2)} + \tfrac{1}{64}a_3^{(2)}),$$

where again $a_2^{(2)}$ and $a_3^{(2)}$ are computed from (3.5) using $a_1^{(2)}$. Hence, $a_1^{(3)} = 5{\cdot}71$.

The successive values of $a_1^{(k)}$ are, therefore,

$$4, \quad 6, \quad 5{\cdot}71. \tag{3.6}$$

Noting that a_1 is also equal to the Nusselt number based on the cylinder diameter (Frankel & Acrivos 1968), it is evident that the third term of the sequence almost coincides with the exact result 5·73.

Of course, no claim is made that the procedure outlined above will converge, let alone that it will converge to the correct answer. In fact, the fourth term in (3.6) is found to equal 5·5. Nevertheless, the technique appears to be useful, because even the second term in the sequence (3.6) provides a highly satisfactory estimate for the Nusselt number.

In view of the close similarity between the cylinder and sphere problems, it appears reasonable therefore to apply this approximate method to (2.11). First, however, we need to derive the form of (2.10) near the surface of the sphere.

Let $\lambda \equiv g(1)+E = 1{\cdot}047+E$. Then, expanding (2.5) and (2.7) about $r = 1$, we obtain, along a streamline,

$$r-1 = \tfrac{5}{18}\lambda^3\{1+\tfrac{5}{6}\lambda^3(\tfrac{25}{36}-\tfrac{5}{2}\sin^2\theta\cos^2\phi)+O(\lambda^6)\} \tag{3.7}$$

and
$$\cos\theta = \tfrac{6}{5}C\lambda^{-2}\{1-\lambda^3(\tfrac{35}{108}+\tfrac{25}{18}\sin^2\theta\cos^2\phi)+O(\lambda^6)\}. \tag{3.8}$$

Also, in view of (3.8), (2.9) becomes

$$C^* = \tfrac{5}{6}\lambda^2\{1+\tfrac{35}{108}\lambda^3+O(\lambda^6)\}. \tag{3.9}$$

Next, to determine the coefficients of (2.10), we transform variables from the conventional spherical co-ordinates (r,θ,ϕ) to (λ, C, ϕ). Noting that $T = T(\lambda, C)$, and making use of (2.5), (2.6), (3.7) and (3.8), we obtain, after a considerable amount of algebra, that

$$\begin{aligned}\nabla^2 T \sim\; & \lambda^{-4}(\partial^2T/\partial\lambda^2)\{\tfrac{36}{25}-\tfrac{10}{3}\lambda^3+12\lambda^3\gamma+O(\lambda^6)\}\\ & -\lambda^{-5}(\partial T/\partial\lambda)\{\tfrac{72}{25}-\tfrac{61}{15}\lambda^3+6\lambda^3\gamma+O(\lambda^6)\}\\ & +\lambda^{-6}C^2(\partial^2T/\partial C^2)\{\tfrac{144}{25}-\tfrac{116}{15}\lambda^3+24\lambda^3\gamma+O(\lambda^6)\}\\ & -\lambda^{-6}C(\partial T/\partial C)\{\tfrac{72}{25}-\tfrac{106}{15}\lambda^3+12\lambda^3\gamma+O(\lambda^6)\}\\ & +\lambda^{-5}C(\partial^2T/\partial\lambda\,\partial C)\{\tfrac{144}{25}-\tfrac{158}{15}\lambda^3+36\lambda^3\gamma+O(\lambda^6)\},\end{aligned} \tag{3.10}$$

where $\gamma \equiv \sin^2\theta\cos^2\phi$. Interestingly enough, of the terms in ∇^2T, expressed in spherical co-ordinates, only $r^{-2}\,\partial(r^2\,\partial T/\partial r)/\partial r$ contributes to (3.10) to the indicated order in λ.

To find the coefficients $\bar{b}_k$ of (2.11), we multiply (3.10) by $r\sin\theta\, d\phi/4\pi u_\phi$ and integrate from 0 to 2π along the streamline (λ, C). Using the solution given by Cox *et al.* (1968),

$$\frac{r\sin\theta}{u_\phi}d\phi = \frac{2d\phi}{r^{-5}+2(1-r^{-5})\cos^2\phi} = 2d\phi\{1-\tfrac{25}{18}\lambda^3\cos 2\phi+O(\lambda^6)\}.$$

However, since, to a first approximation, θ is constant along a streamline in view of (3.8), it is evident that the coefficients of (2.11) will be those of (3.10) except for γ which becomes $\frac{1}{2}\sin^2\theta$. This equation can be rearranged further into a more convenient form by defining a new variable

$$\eta \equiv \tfrac{6}{5}C\lambda^{-2} = \cos\theta\{1+O(\lambda^3)\}. \tag{3.11}$$

Hence, again after some algebraic manipulations, (2.11) becomes finally

$$\begin{aligned} 0 = \lambda^{-4}(\partial^2 T/\partial\lambda^2)\{\tfrac{36}{25}-\tfrac{10}{3}\lambda^3+6\lambda^3(1-\eta^2)+O(\lambda^6)\} \\ -\lambda^{-5}(\partial T/\partial\lambda)\{\tfrac{72}{25}-\tfrac{61}{15}\lambda^3+3\lambda^3(1-\eta^2)+O(\lambda^6)\} \\ +\lambda^{-2}\eta(\partial^2 T/\partial\eta\,\partial\lambda)\{\tfrac{42}{15}-6(1-\eta^2)+O(\lambda^3)\} \\ +O(1)\{\eta^2\,\partial^2 T/\partial\eta^2, \eta\,\partial T/\partial\eta\}, \end{aligned} \tag{3.12}$$

subject to the conditions

$$T = 1 \quad \text{at} \quad \lambda = 0, \qquad T = 0 \quad \text{at} \quad \lambda = g(1) = 1{\cdot}047,$$

$$T \text{ finite} \quad \text{for} \quad 0 \leqslant \lambda \leqslant 1{\cdot}047, \quad -\eta^* \leqslant \eta \leqslant \eta^*,$$

where, in view of (3.9) and (3.11)

$$\eta^* = 1+\tfrac{35}{108}\lambda^3+O(\lambda^6).$$

To solve (3.12) we now resort to the approximate method described earlier in connexion with the corresponding problem of the freely rotating circular cylinder. Thus, in view of (3.12), the expression for T analogous to (3.4) is

$$T = 1-a_1(\eta)\lambda^3+a_2(\eta)\lambda^6+\ldots, \tag{3.13}$$

which, when substituted in (3.12), leads to

$$a_2 = \tfrac{25}{216}\{[-\tfrac{13}{5}+9(1-\eta^2)]\,a_1+[\tfrac{42}{15}-6(1-\eta^2)]\,\eta\,da_1/d\eta\}. \tag{3.14}$$

Therefore, if only two terms in (3.13) are retained and the boundary condition at the limiting stream surface $\lambda = 1{\cdot}047$ is applied,

$$a_1^{(1)} = 1/(1{\cdot}047)^3.$$

On the other hand, if three terms in (3.13) are retained, $a_2^{(1)}$ is computed from (3.14) using $a_1^{(1)}$, and the Euler transformation is employed

$$1 = \tfrac{1}{2}(1{\cdot}047)^3\{\tfrac{3}{2}a_1^{(2)}-\tfrac{1}{2}(1{\cdot}047)^3\,a_2^{(1)}\},$$

from which $\qquad a_1^{(2)} = [4/3(1{\cdot}047)^3]-\tfrac{25}{648}[\tfrac{13}{5}-9(1-\eta^2)].$

Finally, because of (2.12), (3.7), (3.11) and (3.13),

$$N_u = \frac{18}{5}\int_{-1}^{1} a_1\,d\eta;$$

hence the values for N_u computed from $a_1^{(1)}$ and $a_1^{(2)}$ are, successively, equal to 6·27 and 9·31. Their ratio is equal to 1·48 which is almost that of the first two terms of (3.6). Consequently, it appears that a reasonable estimate for the asymptotic Nusselt number is

$$N_u \sim 9{\cdot}31\times\tfrac{1}{6}\times 5{\cdot}73 = 8{\cdot}9 \sim 9.$$

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